7.4 - Operational Properties II

Derivatives of Transforms

The formulas we saw before $\mathscr{L}\{y'\} = sY(s) - y(0)$ and $\mathscr{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$ are transforms of derivatives. Now we will consider *derivatives of transforms*.

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}[F(s)]$$

Example: Evaluate the given Laplace transform.

$\mathscr{L}\{t \sin t\}$

$\mathscr{L}\left\{te^{-3t} \sin t\right\}$

In general, we have

Theorem 7.4.1: Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1,\ 2,\ 3, \ldots$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$.

Example: Use Theorem 7.4.1 to evaluate the given Laplace transform.

$\mathscr{L}\left\{ t^2 \cos t \right\}$

Example: Use the Laplace transform to solve the given initial-value problem. Use the table of Laplace transforms in Appendix C as needed.

$y' - y = te^t \sin t, \qquad y(0) = 0$

$y'' + y = f(t), \qquad y(0) = 1, \quad y'(0) = 0$, where $f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ \sin t, & t \ge \pi/2 \end{cases}$

Convolution

Definition: The **convolution** of f and g is $f * g = \int_0^t f(\tau)g(t-\tau)\, d\tau$

Example: Find $\cos 2t * e^t$.

Example: Find $\mathscr{L}\{\cos 2t * e^t\}$.

In general,

Theorem: Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then $\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$.

Example: Find the Laplace transform of $f * g$ using the Convolution Theorem. Do not evaluate the convolution integral before transforming.

$\mathscr{L}\left\{ t^3 * t\sin 3t \right\}$

$\mathscr{L}\left\{ \int_0^t \sin\tau\cos(t-\tau)\,d\tau \right\}$

$\mathscr{L}\left\{ \int_0^t e^{-\tau}\cos\tau\,d\tau \right\}$

The transform of an integral

When $g(t) = 1$, $\mathscr{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem gives

$\mathscr{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \dfrac{F(s)}{s}$, which in inverse form is $\int_0^t f(\tau)\, d\tau = \mathscr{L}^{-1}\left\{\dfrac{F(s)}{s}\right\}$.

Example: Evaluate the given inverse transform.

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\}$$

Example: Use the Laplace transform to solve the given integral equation.

$$f(t) + 2\int_0^t f(\tau)\cos(t-\tau)\, d\tau = 4e^{-t} + \sin t$$

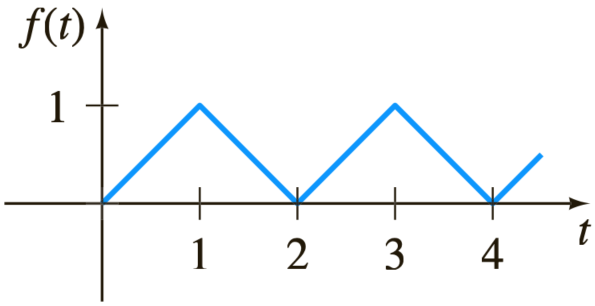
Example: Use the Laplace transform to solve the given integrodifferential equation.

$$\frac{dy}{dt} + 6y(t) + 9\int y(\tau)\, d\tau = 1, \qquad y(0) = 0$$

Theorem: Transform of a Periodic Function

If $f(t)$ is piecewise continuous on $[0,\ \infty)$, of exponential order, and periodic with period T, then $\mathscr{L}\{f(t)\} = \dfrac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$.

Example: Find the Laplace transform of the given periodic function.



triangular wave